

DEFINING RELATIONS OF LOW DEGREE OF INVARIANTS OF TWO 4×4 MATRICES

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ABSTRACT. The trace algebra C_{nd} over a field of characteristic 0 is generated by all traces of products of d generic $n \times n$ matrices, $n, d \geq 2$. Minimal sets of generators of C_{nd} are known for $n = 2$ and 3 for any d and for $n = 4$ and 5 and $d = 2$. The explicit defining relations between the generators are found for $n = 2$ and any d and for $n = 3, d = 2$ only. Defining relations of minimal degree for $n = 3$ and any d are also known. The minimal degree of the defining relations of any homogeneous minimal generating set of C_{42} is equal to 12. Starting with the generating set given recently by Drensky and Sadikova, we have determined all relations of degree ≤ 14 . For this purpose we have developed further algorithms based on representation theory of the general linear group and easy computer calculations with standard functions of Maple.

INTRODUCTION

Let K be any field of characteristic 0. All vector spaces, tensor products, algebras considered in this paper are over K . Let $X_i = (x_{pq}^{(i)})$, $p, q = 1, \dots, n$, $i = 1, \dots, d$, be d generic $n \times n$ matrices. We consider the pure (or commutative) trace algebra C_{nd} generated by all traces of products $\text{tr}(X_{i_1} \cdots X_{i_k})$. It coincides with the algebra of invariants of the general linear group $GL_n = GL_n(K)$ acting by simultaneous conjugation on d matrices of size $n \times n$. The algebra C_{nd} is finitely generated. An upper bound for the degree of the trace monomials sufficient to generate C_{nd} is given in terms of the Nagata-Higman theorem in the theory of PI-algebras. The defining relations of C_{nd} are described by the Razmyslov-Procesi theory [R, P] in the

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language of ideals of the group algebras of symmetric groups. For a background on the algebras of matrix invariants see e.g. [F, DF] and for computational aspects of the theory see [D2].

Explicit minimal sets of generators of C_{nd} are known for $n = 2$ and 3 for any d , and $n = 4$ and 5 for $d = 2$ only. The exact upper bound of the degree $k \leq N(n)$ of the trace polynomials $\text{tr}(X_{i_1} \cdots X_{i_k})$ sufficient to generate C_{nd} is $N(2) = 3$, $N(3) = 6$, and $N(4) = 10$. Even less is known for the defining relations between these minimal sets of generators. For details on the explicit form of the defining relations for $n = 2$, $d \geq 2$ see e.g. [DF]. For $n = 3$, $d = 2$, a minimal generating set of C_{32} consisting of 11 trace monomials of degree ≤ 6 was found by Teranishi [T1]. He also calculated the Hilbert (or Poincaré) series of C_{32} . It follows from his description that, with respect to these generators, C_{32} has a single defining relation of degree 12. The explicit form of the relation was found by Nakamoto [N], over \mathbb{Z} , with respect to a slightly different system of generators. Abeasis and Pittaluga [AP] found a system of generators of C_{3d} , for any $d \geq 2$, in terms of representation theory of the symmetric and general linear groups, in the spirit of its usage in theory of PI-algebras. Aslaksen, Drensky and Sadikova [ADS] gave the defining relation of C_{32} with respect to the generators from [AP]. For $n = 3$ and $d > 2$ the defining relations of C_{3d} seem to be very complicated. Recently, Benanti and Drensky [BD] have shown that for all $d > 2$ the minimal degree of the defining relations of C_{3d} is equal to 7 and have found explicitly these relations with respect to the generators from [AP]. For $d = 3$ they have given also the relations of degree 8, using additional information from the Hilbert series of C_{33} calculated by Berele and Stembridge [BS]. Independently, the defining relations of the algebra C_{33} have been studied in the recent master thesis of Hoge [H]. Using representation theory of general linear groups and computer calculations with Maple, as in [ADS] and [BD], he developed a general algorithm and found the relations of degree 7 and some of the relations of degree 8.

For C_{42} , a set of generators was found by Teranishi [T1, T2] and a minimal set by Drensky and Sadikova [DS], in terms of the approach in [AP]. Djoković [Dj] gave another minimal set of 32 generators of C_{42} consisting of trace monomials only (he found also a minimal set of 173 generators of C_{52}). Any homogeneous minimal generating set $\{u_i \mid i = 1, \dots, 32\}$ of C_{42} consists of g_i elements of degree $i = 1, 2, \dots, 10$,

where

$$(1) \quad \begin{aligned} g_1 = 2, \quad g_2 = 3, \quad g_3 = 4, \quad g_4 = 6, \quad g_5 = 2, \\ g_6 = 4, \quad g_7 = 2, \quad g_8 = 4, \quad g_9 = 4, \quad g_{10} = 1. \end{aligned}$$

Hence C_{42} is isomorphic to the factor algebra $K[y_1, \dots, y_{32}]/I$. Defining $\deg(y_i) = \deg(u_i)$, the ideal I is homogeneous. The comparison of the Hilbert series of C_{42} calculated by Teranishi [T2] (with some typos) and corrected by Berele and Stembridge [BS], with the Hilbert series of $K[y_1, \dots, y_{32}]$ gives that any homogeneous minimal system of generators of the ideal I contains no elements of degree ≤ 11 and 5 elements of degree 12, see [DS]. The purpose of the present paper is to find the explicit form of the defining relations of minimal degree for C_{42} , with respect to the generating set in [DS]. We have performed similar computations also for higher degrees, up to 14. The proofs are based on representation theory of GL_2 combined with computer calculations with Maple and develop further ideas of [ADS, DS]. In particular, we have found a way to write the defining relations in a compact form. Our methods are quite general and can be successfully used for further investigation of generic trace algebras and other algebras close to them.

Having in hand some defining relations of C_{42} , we face the problem what is their meaning. We suggest the following point of view. It is known that the algebra C_{nd} is Cohen-Macaulay. It has a homogeneous system of parameters u_1, \dots, u_p which are algebraically independent and C_{nd} is a finitely generated free $K[u_1, \dots, u_p]$ -module. Here $p = (d-1)n^2 + 1$ is the Krull dimension of C_{nd} . In our case the homogeneous system of parameters of C_{42} consists of 17 of the 32 generators u_i of C_{42} , say u_1, \dots, u_{17} , and the free $K[u_1, \dots, u_{17}]$ -module C_{42} is freely generated by a finite set of products

$$(2) \quad \{u_{18}^{a_{18}} \cdots u_{32}^{a_{32}} \mid (a_{18}, \dots, a_{32}) \in A\}$$

for some set of indices A . The form of the relations of low degree which we have found agrees with the fact that every product $u_{18}^{b_{18}} \cdots u_{32}^{b_{32}}$ can be written as a linear combination of the elements from (2) with coefficients from $K[u_1, \dots, u_{17}]$ and gives some restrictions on the indices (a_{18}, \dots, a_{32}) .

1. PRELIMINARIES

Till the end of the paper we fix $n = 4$ and $d = 2$ and denote by X, Y the two generic 4×4 matrices. We shall denote C_{42} by C . It is a standard trick to replace the generic matrices with generic traceless

matrices. We express X and Y in the form

$$X = \frac{1}{4}\text{tr}(X)e + x, \quad Y = \frac{1}{4}\text{tr}(Y)e + y,$$

where e is the identity 4×4 matrix and x, y are generic traceless matrices. Then

$$(3) \quad C \cong K[\text{tr}(X), \text{tr}(Y)] \otimes C_0,$$

where the algebra C_0 is generated by the traces of products $\text{tr}(z_1 \cdots z_k)$, $z_i = x, y$, $2 \leq k \leq 10$. Hence the problem for the generators and the defining relations of C can be replaced by a similar problem for C_0 .

As in the case of “ordinary” generic matrices, up to similarity we may replace x by a generic traceless diagonal matrix. Although not essential, this results in a simplification from a computational point of view. In fact, one of the worst drawback when computing with traces of polynomials in generic matrices is that these are commutative polynomials with a very high number of monomials. Then, without loss of generality we can fix the two generic traceless matrices as

$$x = \begin{pmatrix} x_{11} & 0 & 0 & 0 \\ 0 & x_{22} & 0 & 0 \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & -(x_{11} + x_{22} + x_{33}) \end{pmatrix},$$

$$y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{44} \\ y_{21} & y_{22} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{33} & y_{34} \\ y_{41} & y_{42} & y_{43} & -(y_{11} + y_{22} + y_{33}) \end{pmatrix}$$

We summarize now the necessary background on representation theory of general linear groups GL_d . We shall state everything for $d = 2$ only. See [M, W] for details on polynomial representations of GL_d and [D1] for their applications to PI-algebras. The group $GL_2 = GL_2(K)$ acts in a canonical way on the vector space with basis $\{x, y\}$ and this action induces a diagonal action on the free associative algebra $K\langle x, y \rangle$:

$$g(z_1 \cdots z_k) = g(z_1) \cdots g(z_k), \quad z_i = x, y, \quad g \in GL_2.$$

The action of GL_2 on $K\langle x, y \rangle$ induces an action on the algebras C and C_0 . For C_0 it is given by

$$g(\text{tr}(z_1 \cdots z_k)) = \text{tr}(g(z_1) \cdots g(z_k)), \quad z_i = x, y, \quad g \in GL_2.$$

The GL_2 -module $K\langle x, y \rangle$ is a direct sum of irreducible polynomial modules, described in terms of partitions $\lambda = (\lambda_1, \lambda_2)$. We denote by $W(\lambda)$ the corresponding GL_2 -module.

The GL_2 -submodules and factor modules W of $K\langle x, y \rangle$ inherit its natural bigrading which counts the entries of x and y in each monomial. We denote by $W^{(p,q)}$ the corresponding homogeneous component of degree p and q in x and y , respectively. The formal power series

$$H(W, t, u) = \sum_{p,q \geq 0} \dim(W^{(p,q)}) t^p u^q$$

is called the Hilbert series of W . The Hilbert series of $W(\lambda)$ is the Schur function $S_\lambda(t, u)$ which, in the case of two variables, has the simple form

$$(4) \quad S_\lambda(t, u) = (tu)^{\lambda_2} (t^{\lambda_1 - \lambda_2} + t^{\lambda_1 - \lambda_2 - 1} u + \dots + tu^{\lambda_1 - \lambda_2 - 1} + u^{\lambda_1 - \lambda_2}).$$

The Hilbert series of W plays the role of its character. The module $W(\lambda)$ participates in W with multiplicity $m(\lambda)$, i.e.,

$$W = \bigoplus (W(\lambda))^{\oplus m(\lambda)}, \quad m(\lambda) \in \mathbb{N} \cup \{0\},$$

if and only if

$$H(W, t, u) = \sum m(\lambda) S_\lambda(t, u).$$

Let $C_0^+ = \omega(C_0)$ be the augmentation ideal of C_0 . It consists of all trace polynomials $f(x, y) \in C_0$ without constant terms, i.e., satisfying the condition $f(0, 0) = 0$. Any minimal system of generators of C_0 lying in C_0^+ forms a basis of the vector space C_0^+ modulo $(C_0^+)^2$. Abeasis and Pittaluga [AP] suggested to fix the minimal system of generators of C_{nd} in such a way that it spans a GL_d -module G . Then C_{nd} is a homomorphic image of the symmetric algebra $K[G] = \text{Sym}(G)$ and the defining relations correspond to the generators of the kernel of the natural homomorphism $K[G] \rightarrow C_{nd}$. Drensky and Sadikova [DS] found that the minimal GL_2 -module of generators of C_{42} is decomposed as

$$\begin{aligned} G = & W(1, 0) \oplus W(2, 0) \oplus W(3, 0) \oplus W(4, 0) \oplus W(2, 2) \\ (5) \quad & \oplus W(3, 2) \oplus W(4, 2) \oplus W(3, 3) \oplus W(4, 3) \\ & \oplus W(5, 3) \oplus W(4, 4) \oplus W(6, 3) \oplus W(5, 5). \end{aligned}$$

Hence the minimal generating GL_2 -module G_0 of C_0 is the direct sum of those modules in (5) which are different from $W(1, 0)$. For $\lambda = (\lambda_1, \lambda_2) \neq (5, 5)$, one may choose as a generator of $W(\lambda_1, \lambda_2) \subset G_0$ the canonical element

$$(6) \quad w_\lambda(x, y) = \text{tr}((xy - yx)^{\lambda_2} x^{\lambda_1 - \lambda_2}).$$

A generator of $W(5, 5)$ may be chosen as

$$(7) \quad w_{(5,5)}(x, y) = \text{tr}((xy - yx)^3(x^2y^2 - xy^2x - yx^2y + y^2x^2)).$$

In [DS] it corresponds to the standard tableau

$$\begin{bmatrix} 1 & 3 & 5 & 7 & 8 \\ 2 & 4 & 6 & 9 & 10 \end{bmatrix}.$$

Since $C_0 \cong K[G_0]/J$ for an ideal J which is also graded, the difference of the Hilbert series of $K[G_0]$ and C_0 gives the Hilbert series of J . By [DS], the Hilbert series of J is

$$\begin{aligned} H(J, t, u) &= H(C_0, t, u) - H(K[G_0], t, u) = (S_{(7,5)}(t, u) + 2S_{(6,6)}(t, u)) \\ &+ (S_{(8,5)}(t, u) + 2S_{(7,6)}(t, u)) + (2S_{(9,5)}(t, u) + 6S_{(8,6)}(t, u) + 2S_{(7,7)}(t, u)) \\ &+ (2S_{(10,5)}(t, u) + 9S_{(9,6)}(t, u) + 7S_{(8,7)}(t, u)) + \cdots. \end{aligned}$$

Hence, the GL_2 -modules R_{12}, R_{13} , and R_{14} of the defining relations of degree 12, 13, and 14 are, respectively,

$$R_{12} = W(7, 5) \oplus 2W(6, 6),$$

$$(8) \quad R_{13} = W(8, 5) \oplus 2W(7, 6),$$

$$R_{14} = 2W(9, 5) \oplus 6W(8, 6) \oplus 2W(7, 7).$$

Any submodule $W(\lambda) = W(\lambda_1, \lambda_2)$ of $K\langle x, y \rangle$ is generated by a unique, up to a multiplicative constant, homogeneous element $w_\lambda(x, y)$ of degree λ_1 and λ_2 in x and y , respectively, called the “highest weight vector” of $W(\lambda)$. It is characterized in the following way, see [DEP, ADF] and [K] for the version which we need. We state it for two variables only. Recall that a linear operator δ on an algebra R is called a derivation if $\delta(uv) = \delta(u)v + u\delta(v)$ for all $u, v \in R$. We define a derivation Δ of $K\langle x, y \rangle$ by putting

$$(9) \quad \Delta(x) = 0, \quad \Delta(y) = x$$

and a linear operator $h \in GL_2$ by $h(x) = x$, $h(y) = x + y$, i.e.,

$$(10) \quad h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The GL_2 -submodules and factor modules of $K\langle x, y \rangle$ are invariant under the action of Δ and we can extend Δ also to tensor products, symmetric algebras, and other constructions with such modules. For example, if $W_1, W_2 \subset K\langle x, y \rangle$, we define Δ on the tensor product $W_1 \otimes W_2$ by $\Delta(w_1 \otimes w_2) = \Delta(w_1) \otimes w_2 + w_1 \otimes \Delta(w_2)$, $w_i \in W_i$.

Lemma 1.1. ([ADF, DEP, K], see also [BD]) *Let Δ and h be defined as in (9) and (10), respectively. The homogeneous polynomial $w_\lambda(x, y) \in K\langle x, y \rangle$ of degree (λ_1, λ_2) is a highest weight vector for some $W(\lambda_1, \lambda_2)$ if and only if $\Delta(w_\lambda(x, y)) = 0$ or, equivalently, $h(w_\lambda(x, y)) = w_\lambda(x, y)$.*

If $W_i \subset K\langle x, y \rangle$, $i = 1, \dots, k$, are k isomorphic copies of $W(\lambda)$ and $w_i \in W_i$ are highest weight vectors, then w_1, \dots, w_k span a vector subspace $V = Kw_1 + \dots + Kw_k$ of $K\langle x, y \rangle$ with the following property. The nonzero elements of V are highest weight vectors of submodules $W(\lambda)$ of the sum $W_1 + \dots + W_k$ and every highest weight vector can be obtained in such a way. The sum $W_1 + \dots + W_k$ is direct if and only if w_1, \dots, w_k are linearly independent. The following statement is a direct consequence of Lemma 1.1.

Corollary 1.2. *If $W(\lambda)$, $\lambda = (\lambda_1, \lambda_2)$, participates with multiplicity $m(\lambda)$ in the GL_2 -submodule W of $K\langle x, y \rangle$, then the vector space of the highest weight vectors $w_\lambda(x, y)$ is an $m(\lambda)$ -dimensional subspace of the homogeneous component $W^{(\lambda_1, \lambda_2)}$ of W . Any basis $\{w_1, \dots, w_{m(\lambda)}\}$ of this subspace generates the direct sum $(W(\lambda))^{\oplus m(\lambda)} \subset W$ as GL_2 -submodule.*

2. ALGORITHMS

For our concrete computations we need the explicit form of the highest weight vectors in the symmetric algebra $K[G_0]$, where $G_0 = G/W(1, 0)$ generates C_0 and G is given in (5). In [ADS, BD, DS] a similar problem was solved by careful study of the symmetric tensor powers $K[W(\lambda)]$ and their tensor products, based on the Littlewood-Richardson rule (or, for $d = 2$, on its partial case, the Young rule) and symmetric tensor powers on the level of [M, Th]. In the present paper we use a simplified approach and work directly in the symmetric algebra $K[G_0]$. (After we had finished the computations we learned that a similar simplification was used independently by Hoge [H].) We define the derivation Δ_1 of $K\langle x, y \rangle$ by

$$(11) \quad \Delta_1(x) = y, \quad \Delta_1(y) = 0$$

and a linear operator $h_1 \in GL_2$ by $h_1(x) = x + y$, $h_1(y) = y$, i.e.,

$$(12) \quad h_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

As in the case of Δ from (9) we extend the action of Δ_1 on GL_2 -modules related with $K\langle x, y \rangle$. The following lemma gives an algorithm which finds a basis of $W(\lambda)$.

Lemma 2.1. *If $\lambda = (a + b, b)$ and $w(x, y) \in W(\lambda) \subset K\langle x, y \rangle$ is a highest weight vector, then the set*

$$(13) \quad \left\{ w, \frac{\Delta_1(w)}{a}, \frac{\Delta_1^2(w)}{a(a-1)}, \dots, \frac{\Delta_1^a(w)}{a(a-1) \cdots 2 \cdot 1} \right\}$$

is a basis of the module $W(\lambda)$. Here Δ_1 is the derivation defined in (11).

Proof. It is well known that, starting with a highest weight vector $w \in W(\lambda)$, the homogeneous components of $h_1(w)$ form a basis of $W(\lambda)$, where $h_1 \in GL_2$ is from (12). Now the proof follows from the fact that, up to a multiplicative constant, $\Delta_1^k(w)$ is equal to the homogeneous component of degree $(a + b - k, b + k)$ of $\varepsilon_1(w)$, where

$$\varepsilon_1 = \exp(\Delta_1) = 1 + \Delta_1/1! + \Delta_1^2/2! + \cdots$$

is the related exponential automorphism of the locally nilpotent derivation Δ_1 , and $h_1 = \exp(\Delta_1)$. \square

Example 2.2. (i) The GL_2 -module $F(a)$ of the forms of degree a in the polynomial algebra $K[x, y]$ in two variables x, y is isomorphic to $W(a, 0)$ and $w = x^a$ is its highest weight vector. Since $\Delta_1(y) = 0$, we obtain

$$\begin{aligned} \Delta_1(w) &= ax^{a-1}y, \Delta_1^2(w) = a(a-1)x^{a-2}y^2, \dots, \\ \Delta_1^{a-1}(w) &= a(a-1) \cdots 2 \cdot xy^{a-1}, \Delta_1^a(w) = a(a-1) \cdots 2 \cdot 1 \cdot y^a, \Delta_1^{a+1}(w) = 0. \end{aligned}$$

Hence Lemma 2.1 gives the basis of $F(a)$

$$\{x^a, x^{a-1}y, \dots, xy^{a-1}, y^a\}.$$

(ii) Consider the submodules of G_0 in (5). The basis of $W(4, 0)$ consists of the highest weight vector

$$w = \text{tr}(x^4),$$

$$\begin{aligned} \frac{\Delta_1(w)}{4} &= \frac{1}{4} \text{tr}(yx^3 + xyx^2 + x^2yx + x^3y) = \text{tr}(x^3y), \\ \frac{\Delta_1^2(w)}{4 \cdot 3} &= \frac{1}{3} \text{tr}((yx^2 + xyx + x^2y)y) = \frac{1}{3} (2 \text{tr}(x^2y^2) + \text{tr}(xyxy)), \\ \frac{\Delta_1^3(w)}{4 \cdot 3 \cdot 2} &= \text{tr}(xy^3), \\ \frac{\Delta_1^4(w)}{4 \cdot 3 \cdot 2 \cdot 1} &= \text{tr}(y^4). \end{aligned}$$

The basis of $W(5, 3) \subset G_0$ consists of

$$\begin{aligned} w &= \text{tr}((xy - yx)^3 x^2), \\ \frac{\Delta_1(w)}{2} &= \frac{1}{2} \text{tr}((xy - yx)^3 (yx + xy)), \end{aligned}$$

$$\frac{\Delta_1^2(w)}{2 \cdot 1} = \text{tr}((xy - yx)^3 y^2).$$

Note that we make use of the fact that the trace of a product does not change under a cyclic permutation of its factors.

Applying Corollary 1.2 we obtain the following algorithm which is in the base of our further computations.

Algorithm 2.3.

Input. A partition $\lambda = (\lambda_1, \lambda_2)$ and a system of highest weight vectors $w_i \in W_i$, $i = 1, \dots, k$, where each W_i is an irreducible GL_2 -submodule of $K\langle x, y \rangle$.

Output. A basis of the vector space of highest weight vectors $w_\lambda(x, y)$ in the symmetric algebra $K[W]$ of the direct sum $W = W_1 \oplus \dots \oplus W_k$.

Step 1. Applying Lemma 2.1, find homogeneous bases $\{u_{i0}, \dots, u_{ia_i}\}$ of the modules W_i , $i = 1, \dots, k$.

Step 2. In $K[W]$, form all products

$$(14) \quad w_p = \prod_{i=1}^k \prod_{j=0}^{a_i} u_{ij}^{r_{ij}}, \quad p = 1, \dots, P,$$

$$(15) \quad v_q = \prod_{i=1}^k \prod_{j=0}^{a_i} u_{ij}^{s_{ij}}, \quad q = 1, \dots, Q,$$

which are of degree (λ_1, λ_2) and $(\lambda_1 + 1, \lambda_2 - 1)$, respectively. Present each $\Delta(w_p)$ in the form

$$\Delta(w_p) = \sum_{q=1}^Q \alpha_{qp} v_q, \quad \alpha_{qp} \in K.$$

Step 3. Consider the element

$$w = \sum_{p=1}^P \xi_p w_p,$$

with unknown coefficients $\xi_p \in K$. Calculate

$$\Delta(w) = \sum_{q=1}^Q \left(\sum_{p=1}^P \alpha_{qp} \xi_p \right) v_q.$$

Step 4. Solve the homogeneous linear system

$$(16) \quad \sum_{p=1}^P \alpha_{qp} \xi_p = 0, \quad q = 1, \dots, Q,$$

whose equations are obtained from the equation $\Delta(w) = 0$.

Step 5. Any basis

$$\{\Xi_r = (\xi_1^{(r)}, \dots, \xi_P^{(r)}) \mid r = 1, \dots, s\}$$

of the vector space of solutions of the system gives rise to a basis of the space of highest weight vectors.

Remark 2.4. Instead of solving one big system (16), we may solve several systems of smaller size. Let $W_i = W(\nu^{(i)})$ for some partition $\nu^{(i)}$. For each m_1, \dots, m_k such that

$$\sum_{i=1}^k m_i |\nu_i| = |\lambda|$$

the vector space $V(m_1, \dots, m_k)$ spanned on those elements w_p from (14) with

$$\sum_{j=0}^{a_i} r_{ij} = m_i, \quad i = 1, \dots, k,$$

is a GL_2 -submodule of $K[W]$. Since

$$K[W] = K[W_1 \oplus \dots \oplus W_k] \cong K[W_1] \otimes \dots \otimes K[W_k],$$

we derive that

$$(17) \quad V(m_1, \dots, m_k) \cong W_1^{\otimes s m_1} \otimes \dots \otimes W_k^{\otimes s m_k},$$

where $W_i^{\otimes s m_i}$ is the m_i -th symmetrized tensor power of W_i . The sum of all $V(m_1, \dots, m_k)$ is direct, and we may choose a basis of the vector space of the λ -highest weight vectors in $K[W]$ as the union of the corresponding bases in $V(m_1, \dots, m_k)$. If $k > 1$, the homogeneous linear systems corresponding to $V(m_1, \dots, m_k)$ are simpler than the whole system (16) for most of the λ .

Obvious modifications of Algorithm 2.3 give the highest weight vectors in other situations. For example, let W_1 and W_2 have homogeneous bases $\{u_0, u_1, \dots, u_p\}$ and $\{v_0, v_1, \dots, v_q\}$, respectively. If we want to find the highest weight vectors in the tensor product $W_1 \otimes W_2$, we have to solve the homogeneous linear system obtained from the equation

$$\Delta \left(\sum \xi_{ij} u_i \otimes v_j \right) = \sum \xi_{ij} (\Delta(u_i) \otimes v_j + u_i \otimes \Delta(v_j)) = 0,$$

where the sum is on all i, j such that $u_i \otimes v_j$ is homogeneous of degree (λ_1, λ_2) .

Remark 2.5. If we want to find only the multiplicity of $W(\lambda)$ in $K[W]$, where $W = W_1 \oplus \cdots \oplus W_k$, we can proceed in the following way. If $W_i = W(\nu^{(i)})$, then the Hilbert series of W is a sum of Schur functions,

$$H(W, t, u) = \sum_{i=1}^k S_{\nu^{(i)}}(t, u) = \sum a_{bc} t^b u^c, \quad a_{bc} \in \mathbb{N} \cup \{0\}.$$

Hence the Hilbert series of $K[W]$ is

$$H(K[W], t, u) = \prod_{b,c} \frac{1}{(1 - t^b u^c)^{a_{bc}}} = \sum_{p,q} h(p, q) t^p u^q, \quad h(p, q) \in \mathbb{N} \cup \{0\}.$$

By (4) the Schur function $S_\mu(t, u)$ contains the summand $t^{\lambda_1} u^{\lambda_2}$ if and only if $\mu_1 + \mu_2 = \lambda_1 + \lambda_2$ and $\mu_1 \geq \lambda_1$. This easily implies that the multiplicity of $W(\lambda)$ is given by the formula

$$(18) \quad m(\lambda) = h(\lambda_1, \lambda_2) - h(\lambda_1 + 1, \lambda_2 - 1).$$

Similarly, if we want to find the multiplicity of $W(\lambda)$ in the tensor product $W_1 \otimes \cdots \otimes W_k$, $W_i = W(\nu^{(i)})$, $i = 1, \dots, k$, we have to present the product of the corresponding Schur functions in the form

$$\prod_{i=1}^k S_{\nu^{(i)}}(t, u) = \sum_{p,q} h(p, q) t^p u^q, \quad h(p, q) \in \mathbb{N} \cup \{0\},$$

and to obtain the multiplicity of $W(\lambda)$ by the formula (18).

We want now to give a compact form for the highest weight vectors of the tensor products $V(m_1, \dots, m_k)$ defined in (17). We fix an order on the summands W_i in the decomposition of $G_0 = G/W(1, 0)$ given in (5). We put:

$$(19) \quad \begin{aligned} W_1 &= W(2, 0), & W_2 &= W(3, 0), & W_3 &= W(4, 0), \\ W_4 &= W(2, 2), & W_5 &= W(3, 2), & W_6 &= W(4, 2), \\ W_7 &= W(3, 3), & W_8 &= W(4, 3), & W_9 &= W(5, 3), \\ W_{10} &= W(4, 4), & W_{11} &= W(6, 3), & W_{12} &= W(5, 5). \end{aligned}$$

For each $W_i = W(\lambda)$ we fix a highest weight vector $w_i = w_\lambda(x, y)$ given in (6) and (7). Rewriting $\lambda = (\lambda_1, \lambda_2)$ in the form $\lambda = (a + b, b)$ we assume that $W_i = W(a_i + b_i, b_i)$. The GL_2 -module $W(a_i + b_i, b_i)$ is isomorphic to the tensor product $\det^{b_i} \otimes W(a_i, 0)$, where \det^{b_i} is the one-dimensional GL_2 -module with GL_2 -action defined by

$$g(v) = (\det(g))^{b_i} \cdot v, \quad g \in GL_2, v \in \det^{b_i}.$$

The module $W(a_i, 0)$ has a natural realization as the module $F(a_i)$ of the forms of degree a_i in two variables x_i, y_i . We fix a nonzero element of \det^{b_i} and denote it by $t_i^{b_i}$. Omitting the symbol \otimes for the tensor product, $\det^{b_i} \otimes F(a_i)$ has a basis

$$\{t_i^{b_i} x_i^{a_i}, t_i^{b_i} x_i^{a_i-1} y_i, \dots, t_i^{b_i} x_i y_i^{a_i-1}, t_i^{b_i} y_i^{a_i}\}$$

with action of GL_2 defined by

$$g(t_i^{b_i} x_i^j y_i^{a_i-j}) = (\det(g))^{b_i} t_i^{b_i} (g(x_i))^j (g(y_i))^{a_i-j}, \quad g \in GL_2.$$

Using the highest weight vector w_i of $W(a_i + b_i, b_i)$ from (6) and (7), we fix the GL_2 -module isomorphism

$$(20) \quad \varphi_i : \det^{b_i} \otimes F(a_i) \rightarrow W_i = W(a_i + b_i, b_i)$$

which sends $t_i^{b_i} x_i^{a_i}$ to $w_i(x, y)$. The concrete form of the image of $t_i^{b_i} x_i^j y_i^{a_i-j}$ in $W(a_i + b_i, b_i)$ can be obtained applying Lemma 2.1. For the derivation Δ_1 from (11),

$$\Delta_1(\det^{b_i}) = 0,$$

$$\frac{1}{j} \Delta_1(x_i^j y_i^{a_i-j}) = x_i^{j-1} y_i^{a_i+1-j}$$

and we define recursively

$$\varphi_i(t_i^{b_i} x_i^{a_i}) = w_i(x, y),$$

$$\varphi_i(t_i^{b_i} x_i^{j-1} y_i^{a_i+1-j}) = \frac{1}{j} \Delta_1(\varphi_i(t_i^{b_i} x_i^j y_i^{a_i-j})), \quad j = a_i, a_i - 1, \dots, 2, 1.$$

For example, if $\lambda = (6, 3)$, then $W(\lambda) = W_{11}$ in the notation of (19),

$$\varphi_{11}(t_{11}^3 x_{11}^3) = w_{11}(x, y) = \text{tr}([x, y]^3 x^3),$$

$$\varphi_{11}(t_{11}^3 x_{11}^2 y_{11}) = \frac{1}{3} \Delta_1(w_{11}) = \frac{1}{3} \text{tr}([x, y]^3 (yx^2 + xyx + x^2y)),$$

$$\varphi_{11}(t_{11}^3 x_{11} y_{11}^2) = \frac{1}{3} \text{tr}([x, y]^3 (y^2x + yxy + xy^2)),$$

$$\varphi_{11}(t_{11}^3 y_{11}^3) = \text{tr}([x, y]^3 y^3).$$

Now we extend the GL_2 -module isomorphisms φ_i to the symmetric algebras. Let

$$(21) \quad \Phi : K \left[\bigoplus_{i=1}^{12} \det^{b_i} \otimes F(a_i) \right] \rightarrow K[G_0]$$

be defined by

$$\Phi \left(\prod_{i=1}^{12} \prod_{j=0}^{a_i} (t_i^{b_i} x_i^j y_i^{a_i-j})^{c_{ij}} \right) = \prod_{i=1}^{12} \prod_{j=0}^{a_i} (\varphi_i(t_i^{b_i} x_i^j y_i^{a_i-j}))^{c_{ij}}, \quad c_{ij} \geq 0.$$

In order to avoid the confusion and to distinguish e.g. $(x_1 y_1)^2 = (x_1 y_1) \otimes (x_1 y_1)$ and $(x_1^2)(y_1^2) = x_1^2 \otimes y_1^2$ in $F(2)^{\otimes s^2}$, in the summands where $\sum_{j=0}^{a_i} c_{ij} > 1$, we shall denote the elements $x_i^j y_i^{a_i-j}$ by $z_i^{(j, a_i-j)}$. Hence, instead of $(x_1 y_1)^2 = (x_1 y_1) \otimes (x_1 y_1)$ and $x_1^2 y_1^2 = x_1^2 \otimes y_1^2$ we shall write $(z_1^{(1,1)})^2$ and $(z_1^{(2,0)})(z_1^{(0,2)})$, respectively. There is no confusion using t_i because

$$W(a_i + b_i, b_i)^{\otimes s m_i} \cong \det^{b_i m_i} \otimes F(a_i)^{\otimes s m_i}.$$

For example, using the notation of (17) and (19) one has

$$\begin{aligned} V(1, 2, 1, 0, \dots, 0) &= W_1 \otimes W_2^{\otimes s^2} \otimes W_3 \\ &= W(2, 0) \otimes W(3, 0)^{\otimes s^2} \otimes W(4, 0), \\ \varphi_1(y_1^2) &= \text{tr}(y^2), \\ \varphi_2(x_2^3) &= \varphi_2(z_2^{(3,0)}) = \text{tr}(x^3), \quad \varphi_2(x_2 y_2^2) = \varphi_2(z_2^{(1,2)}) = \text{tr}(x y^2), \\ \varphi_3(x_3^3 y_3) &= \text{tr}(x^3 y), \\ \Phi(y_1^2(z_2^{(3,0)})(z_2^{(1,2)})x_3^3 y_3) &= \text{tr}(y^2) \text{tr}(x^3) \text{tr}(x y^2) \text{tr}(x^3 y). \end{aligned}$$

For

$$\begin{aligned} V(2, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0) &= W_1^{\otimes s^2} \otimes W_2 \otimes W_5 \\ &= W(2, 0)^{\otimes s^2} \otimes W(3, 0) \otimes W(3, 2), \\ \varphi_1(x_1 y_1) &= \varphi_1(z_1^{(1,1)}) = \text{tr}(x y), \\ \varphi_1(y_1^2) &= \varphi_1(z_1^{(0,2)}) = \text{tr}(y^2), \\ \varphi_2(x_2 y_2^2) &= \text{tr}(x y^2), \\ \varphi_5(t_5^2 x_5) &= \text{tr}([x, y]^2 x), \\ \Phi((z_1^{(1,1)})(z_1^{(0,2)})x_2 y_2^2 t_5^2 x_5) &= \text{tr}(x y) \text{tr}(y^2) \text{tr}(x y^2) \text{tr}([x, y]^2 x). \end{aligned}$$

For

$$\begin{aligned} V(2, 0, 0, 2, 0, \dots, 0) &= W_1^{\otimes s^2} \otimes W_4^{\otimes s^2} = W(2, 0)^{\otimes s^2} \otimes W(2, 2)^{\otimes s^2}, \\ \varphi_1((z_1^{(2,0)})(z_1^{(0,2)})) &= \text{tr}(x^2) \text{tr}(y^2), \\ \varphi_4(t_4^4) &= \text{tr}^2([x, y]^2), \\ \Phi((z_1^{(2,0)})(z_1^{(0,2)})t_4^4) &= \text{tr}(x^2) \text{tr}(y^2) \text{tr}^2([x, y]^2). \end{aligned}$$

3. COMPUTATIONS AND RESULTS

We shall explain now the computations for degree 12. From (8) we see that it is sufficient to consider the cases $\lambda = (7, 5)$ and $\lambda = (6, 6)$ only. First, we use Algorithm 2.3 to find the highest weight vectors $w_\lambda(x, y) \in K[G_0]$. Applying Step 1 of the algorithm we find bases of the submodules W_1, \dots, W_{12} of G_0 . By Step 2, we form all products (14) and (15) of degree (λ_1, λ_2) and $(\lambda_1 + 1, \lambda_2 - 1)$, respectively.

For $\lambda = (7, 5)$ we obtain that $P = 155$ and $Q = 119$, i.e., there are 155 products (14) of degree $(7, 5)$ and 119 products (15) of degree $(8, 4)$. Applying Steps 3 and 4 we compute that the system (16) has $s = 36$ linearly independent solutions which give rise to 36 linearly independent highest weight vectors. Hence $W(7, 5)$ participates with multiplicity 36 in $K[G_0]$. We call these 36 highest weight vectors w_1, \dots, w_{36} .

For $\lambda = (6, 6)$ the corresponding data are $P = 185$, $Q = 155$ and the number of the linear independent highest weight vectors $w_{(6,6)}(x, y) \in K[G_0]$ is $s = 30$.

The next step of the computations is to find the highest weight vectors of the GL_2 -modules $W(\lambda) \subset R_{12}$ of the defining relations of degree 12. For $\lambda = (7, 5)$ we proceed in the following way. We form the trace polynomial in $K[G_0]$

$$(22) \quad w = \sum_{i=1}^{36} \zeta_i w_i,$$

where w_i are the 36 linearly independent highest weight vectors corresponding to the submodules $W(7, 5)$ of $K[G_0]$ and ζ_i are unknown coefficients. Then we evaluate w on the generic traceless 4×4 matrices x and y and obtain

$$(23) \quad w(x, y) = \sum_{i=1}^{36} \zeta_i w_i(x, y) = \sum_{p,q=1}^4 \sum_{i=1}^{36} \zeta_i w_i^{(p,q)}(x, y) e_{pq},$$

where the (p, q) -entry $w_i^{(p,q)}(x, y)$ of $w_i(x, y)$ is a homogeneous polynomial of degree 12 in the entries $x_{aa}, y_{b_1 b_2}$ of x and y . We require that $w(x, y) = 0$ which is equivalent to

$$w^{(p,q)}(x, y) = \sum_{i=1}^{36} \zeta_i w_i^{(p,q)}(x, y) = 0, \quad p, q = 1, 2, 3, 4.$$

We rewrite the relations $w^{(p,q)}(x, y) = 0$ in the form

$$w^{(p,q)}(x, y) = \sum_{c,d} \alpha_{cd}^{(p,q)}(\zeta_1, \dots, \zeta_{36}) \prod_a x_{aa}^{c_a} \prod_{b_1, b_2} y_{b_1 b_2}^{d_{b_1 b_2}} = 0.$$

Since the coefficients $\alpha_{cd}^{(p,q)}$ are equal to 0, we obtain a homogeneous linear system

$$(24) \quad \alpha_{cd}^{(p,q)}(\zeta_1, \dots, \zeta_{36}) = 0$$

with unknowns $\zeta_1, \dots, \zeta_{36}$. The solutions of the system give rise to the highest weight vectors which generate the submodules $W(7, 5)$ of the GL_2 -module R_{12} of defining relations of degree 12. The result of the computations is that the system has a unique nonzero solution which we shall give explicitly soon.

Similar computations for $\lambda = (6, 6)$ give that the system, which corresponds to (24) in this case, has two linearly independent solutions. One of them is relatively simple:

$$(25) \quad v'_{(6,6)} = 3u_1(x, y) + 4u_2(x, y) + 6u_3(x, y) = 0,$$

where

$$\begin{aligned} u_1 &= \text{tr}(x^2)\text{tr}(y^3)\text{tr}([x, y]^3x) - \text{tr}(y^2)\text{tr}(xy^2)\text{tr}([x, y]^3y) \\ &\quad - 2\text{tr}(xy)\text{tr}(xy^2)\text{tr}([x, y]^3x) + 2\text{tr}(xy)\text{tr}(x^2y)\text{tr}([x, y]^3y) \\ &\quad + \text{tr}(y^2)\text{tr}(x^2y)\text{tr}([x, y]^3x) - \text{tr}(y^2)\text{tr}(x^3)\text{tr}([x, y]^3y), \\ u_2 &= -\text{tr}(y^3)\text{tr}([x, y]^3x^3) + 3\text{tr}(xy^2)\text{tr}([x, y]^3(yx^2 + xyx + x^2y)) \\ &\quad - 3\text{tr}(x^2y)\text{tr}([x, y]^3(y^2x + yxy + xy^2)) + \text{tr}(x^3)\text{tr}([x, y]^3y^3), \\ u_3 &= -\text{tr}([x, y]^2x)\text{tr}([x, y]^3y) + \text{tr}([x, y]^2y)\text{tr}([x, y]^3x). \end{aligned}$$

Applying the isomorphism Φ from (21) we rewrite u_1, u_2, u_3 as

$$u_1 = \Phi(-(x_1y_2 - y_1x_2)^2(x_2y_8 - y_2x_8)t_8^3).$$

$$u_2 = \Phi((x_2y_{11} - y_2x_{11})^3t_{11}^3),$$

$$u_3 = \Phi(-(x_5y_8 - y_5x_8)t_5^2t_8^3),$$

Hence (25) has the form

$$(26) \quad \begin{aligned} v'_{(6,6)} &= \Phi(-3(x_1y_2 - y_1x_2)^2(x_2y_8 - y_2x_8)t_8^3 \\ &\quad + 4(x_2y_{11} - y_2x_{11})^3t_{11}^3 - 6(x_5y_8 - y_5x_8)t_5^2t_8^3) = 0. \end{aligned}$$

In the same notation the only solution for the case $\lambda = (7, 5)$ is
(27)

$$\begin{aligned}
v_{(7,5)} = & \Phi(-6z_1^{(2,0)}(z_1^{(2,0)}z_1^{(0,2)} - (z_1^{(1,1)})^2)t_7^3 \\
& - 4((z_1^{(2,0)}z_1^{(0,2)} - 6(z_1^{(1,1)})^2)x_9^2 + 10z_1^{(2,0)}z_1^{(1,1)}x_9y_9 - 5(z_1^{(2,0)})^2y_9^2)t_9^3 \\
& + 4(x_1y_2 - y_1x_2)x_2((x_1y_2 + y_1x_2)x_8 - 2x_1x_2y_8)t_8^3 \\
& + 16(x_1y_3 - y_1x_3)^2x_3^2t_7^3 - x_1^2t_4^2t_7^3 + 8x_1^2t_{12}^5 \\
& + 28(z_2^{(3,0)}z_2^{(1,2)} - (z_2^{(2,1)})^2)t_7^3 - 48x_2x_{11}(x_2y_{11} - y_2x_{11})^2t_{11}^3 \\
& - 48x_3^2(x_3y_9 - y_3x_9)^2t_9^3 - 16x_9^2t_4^2t_9^3 \\
& - 24x_5x_8t_5^2t_8^3 + 4x_6^2t_6^2t_7^3) = 0
\end{aligned}$$

and the second relation for $\lambda = (6, 6)$ is

(28)

$$\begin{aligned}
v''_{(6,6)} = & \Phi(-108(z_1^{(2,0)} z_1^{(0,2)} - (z_1^{(1,1)})^2)^3 \\
& + 216(z_1^{(2,0)} z_1^{(0,2)} - (z_1^{(1,1)})^2)(x_3^2 z_1^{(0,2)} - 2x_3 y_3 z_1^{(1,1)} + y_3^2 z_1^{(2,0)})^2 \\
& - 180(z_1^{(2,0)} z_1^{(0,2)} - (z_1^{(1,1)})^2)^2 t_4^2 \\
& - 12(54 z_1^{(2,0)} (z_1^{(1,1)})^2 (z_2^{(1,2)})^2 + 12(z_1^{(2,0)})^2 z_1^{(0,2)} z_2^{(2,1)} z_2^{(0,3)} \\
& + 30 z_1^{(2,0)} (z_1^{(1,1)})^2 z_2^{(2,1)} z_2^{(0,3)} - 42(z_1^{(2,0)})^2 z_1^{(1,1)} z_2^{(1,2)} z_2^{(0,3)} \\
& - 72 z_1^{(2,0)} z_1^{(1,1)} z_1^{(0,2)} z_2^{(2,1)} z_2^{(1,2)} + 9(z_1^{(2,0)})^2 z_1^{(0,2)} (z_2^{(1,2)})^2 \\
& - 12 z_1^{(2,0)} z_1^{(1,1)} z_1^{(0,2)} z_2^{(3,0)} z_2^{(0,3)} - 42 z_1^{(1,1)} (z_1^{(0,2)})^2 z_2^{(3,0)} z_2^{(2,1)} \\
& + 54(z_1^{(1,1)})^2 z_1^{(0,2)} (z_2^{(2,1)})^2 - 54(z_1^{(1,1)})^3 z_2^{(2,1)} z_2^{(1,2)} \\
& + 30(z_1^{(1,1)})^2 z_1^{(0,2)} z_2^{(3,0)} z_2^{(1,2)} + 7(z_1^{(2,0)})^3 (z_2^{(0,3)})^2 \\
& + 7(z_1^{(0,2)})^3 (z_2^{(3,0)})^2 + 9 z_1^{(2,0)} (z_1^{(0,2)})^2 (z_2^{(2,1)})^2 \\
& - 2(z_1^{(1,1)})^3 z_2^{(3,0)} z_2^{(0,3)} + 12 z_1^{(2,0)} (z_1^{(0,2)})^2 z_2^{(3,0)} z_2^{(1,2)}) \\
& + 216(z_1^{(2,0)} z_1^{(0,2)} - (z_1^{(1,1)})^2)(z_1^{(0,2)} x_6^2 - 2z_1^{(1,1)} x_6 y_6 + z_1^{(2,0)} y_6^2) t_6^2 \\
& + 432(z_1^{(0,2)} x_2^2 - 2z_1^{(1,1)} x_2 y_2 \\
& + z_1^{(2,0)} y_2^2)(-z_1^{(0,2)} x_2 x_5 + z_1^{(1,1)}(x_2 y_5 + x_5 y_2) - z_1^{(2,0)} y_2 y_5) t_5^2 \\
& - 432(-2(z_1^{(2,0)})^2((z_3^{(1,3)})^2 - z_3^{(2,2)} z_3^{(0,4)}) \\
& - 4z_1^{(2,0)} z_1^{(1,1)}(z_3^{(3,1)} z_3^{(0,4)} - z_3^{(2,2)} z_3^{(1,3)}) \\
& - z_1^{(2,0)} z_1^{(0,2)}((z_3^{(2,2)})^2 - z_3^{(4,0)} z_3^{(0,4)}) \\
& (z_1^{(1,1)})^2(-5(z_3^{(2,2)})^2 + z_3^{(4,0)} z_3^{(0,4)} + 4z_3^{(3,1)} z_3^{(1,3)}) \\
& - 4z_1^{(1,1)} z_1^{(0,2)}(z_3^{(4,0)} z_3^{(1,3)} - z_3^{(3,1)} z_3^{(2,2)}) \\
& + 2(z_1^{(0,2)})^2(z_3^{(4,0)} z_3^{(2,2)} - (z_3^{(3,1)})^2)) \\
& + 216(z_1^{(0,2)} x_3^2 - 2z_1^{(1,1)} x_3 y_3 + z_1^{(2,0)} y_3^2) t_4^2 \\
& + 33(z_1^{(2,0)} z_1^{(0,2)} - (z_1^{(1,1)})^2) t_4^4 \\
& + 36(-z_2^{(0,3)} x_3^3 + 3z_2^{(1,2)} x_3^2 y_3 - 3z_2^{(2,1)} x_3 y_3^2 + z_2^{(3,0)} y_3^3)(-x_1^2 x_3 z_2^{(0,3)} \\
& + x_1(x_1 y_3 + 2y_1 x_3) z_2^{(1,2)} - y_1(y_1 x_3 + 2x_1 y_3) z_2^{(2,1)} + y_1^2 y_3 z_2^{(3,0)}) \\
& + 45(x_1^2(z_2^{(2,1)} z_2^{(0,3)} - (z_2^{(1,2)})^2) \\
& - x_1 y_1(z_2^{(3,0)} z_2^{(0,3)} - z_2^{(2,1)} z_2^{(1,2)}) + y_1^2(z_2^{(3,0)} z_2^{(1,2)} - (z_2^{(2,1)})^2)) t_4^2 \\
& - 108(x_1 y_3 - y_1 x_3)^2 (x_3 y_6 - y_3 x_6) t_6^2 \\
& + 9(x_1 y_6 - y_1 x_6)^2 t_4^2 t_6^2 - 108(x_1 y_5 - y_1 x_5)^2 t_5^4 \\
& - 9(4(z_2^{(2,1)})^3 z_2^{(0,3)} - 6z_2^{(3,0)} z_2^{(1,2)} z_2^{(2,1)} z_2^{(0,3)} + (z_2^{(3,0)})^2 (z_2^{(0,3)})^2 \\
& + 4z_2^{(3,0)} (z_2^{(1,2)})^3 - 3(z_2^{(2,1)})^2 (z_2^{(1,2)})^2) \\
& - 144(x_2 y_3 - y_2 x_3)^3 (x_3 y_5 - y_3 x_5) t_5^2 \\
& - 108((z_2^{(2,1)} z_2^{(0,3)} - (z_2^{(1,2)})^2) x_6^2 - (z_2^{(3,0)} z_2^{(0,3)}) \\
& - z_2^{(2,1)} z_2^{(1,2)}) x_6 y_6 + (z_2^{(3,0)} z_2^{(1,2)} - (z_2^{(2,1)})^2) y_6^2) t_6^2 \\
& + 432(-(z_3^{(3,1)})^2 z_3^{(0,4)} + 2z_3^{(3,1)} z_3^{(2,2)} z_3^{(1,3)} \\
& - (z_3^{(2,2)})^3 - z_3^{(4,0)} (z_3^{(1,3)})^2 + z_3^{(4,0)} z_3^{(2,2)} z_3^{(0,4)}) \\
& - 36(z_3^{(4,0)} z_3^{(0,4)} - 4z_3^{(1,3)} z_3^{(3,1)} + 3(z_3^{(2,2)})^2) t_4^2 \\
& + 14t_4^6 - 36t_4^2 t_{10}^4 - 108(z_6^{(2,0)} z_6^{(0,2)} - (z_6^{(1,1)})^2) t_6^4 + 24t_7^6 = 0.
\end{aligned}$$

We state the results as a theorem.

Theorem 3.1. *The defining relations of degree 12 of the pure trace algebra generated by two traceless 4×4 generic matrices form a GL_2 -module isomorphic to $W(7, 5) \oplus 2W(6, 6)$. The corresponding highest weight vectors*

$$v_{(7,5)} = 0, \quad v'_{(6,6)} = 0, \quad v''_{(6,6)} = 0$$

are given in (27), (26), and (28).

The computational results for degree 13 and 14 are quite long expressions and we shall discuss them in the next section. Here for each λ corresponding to a defining relation in (8) we give only the numbers P and Q of the vectors (14) and (15), the number s of linearly independent solutions of the system (16) and the multiplicity r of $W(\lambda)$ from (8):

$$\begin{aligned}
 \lambda = (8, 5) : \quad & P = 203, \quad Q = 136, \quad s = 67, \quad r = 1, \\
 \lambda = (7, 6) : \quad & P = 252, \quad Q = 203, \quad s = 49, \quad r = 2, \\
 (29) \quad \lambda = (9, 5) : \quad & P = 284, \quad Q = 188, \quad s = 96, \quad r = 2, \\
 \lambda = (8, 6) : \quad & P = 390, \quad Q = 284, \quad s = 106, \quad r = 6, \\
 \lambda = (7, 7) : \quad & P = 418, \quad Q = 390, \quad s = 28, \quad r = 2.
 \end{aligned}$$

4. CONCLUSIONS

The homogeneous system of parameters of C_{42} found by Teranishi [T1, T2] contains all traces of degree ≤ 4 and two elements of degree $(4, 2)$ and $(2, 4)$, respectively. If in the system of Teranishi we remove $\text{tr}(X)$, $\text{tr}(Y)$ and replace X, Y with x, y , respectively, we obtain a homogeneous system of parameters of the algebra C_0 generated by the generic traceless 4×4 matrices x and y . Following [DS], we can choose for a homogeneous system of parameters of C_0 any 13 trace polynomials which form a K -basis of

$$W(2, 0) \oplus W(3, 0) \oplus W(4, 0) \oplus W(2, 2) = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \subset G_0$$

and two more trace polynomials

$$\text{tr}((xy - yx)^2 x^2), \text{tr}((xy - yx)^2 y^2) \in W(4, 2) = W_6 \subset G_0.$$

Hence in (2) we can choose for u_{18}, \dots, u_{32} any basis of the direct sum from (19)

$$W_5 \oplus W_7 \oplus W_8 \oplus W_9 \oplus W_{10} \oplus W_{11} \oplus W_{12}$$

and the only homogeneous polynomial of degree $(3, 3)$ in $W(4, 2) = W_6$. Using Lemma 2.1 we construct homogeneous bases $\{u_{i0}, \dots, u_{ia_i}\}$ of the modules $W_i = W(a_i + b_i, b_i)$ from (19). In this notation, we fix the homogeneous system of parameters of C_0 consisting of

$$(30) \quad \{u_{ij}, u_{60}, u_{62} \mid i = 1, 2, 3, 4, j = 0, 1, \dots, a_i\}$$

and complete it to a system of generators of C_0 by

$$(31) \quad \{u_{ij}, u_{61} \mid i = 5, 7, 8, 9, 10, 11, 12, j = 0, 1, \dots, a_i\}.$$

It is easy to see that the finitely generated free S -module C_0 , where

$$(32) \quad S = K[u_{ij} \mid i = 1, 2, 3, 4, j = 0, 1, \dots, a_i, i = 6, j = 0, 2],$$

has a basis of the form

$$(33) \quad B = \left\{ \prod_{i,j} u_{ij}^{b_{ij}} \mid i = 5, 6, \dots, 12 \right\},$$

where the u_{ij} 's are from (31) and the b_{ij} 's belong to some set of indices. Hence every product of elements from (31) can be presented as a linear combination of elements in B from (33) with coefficients in S . We shall use the defining relations of degree 12, 13, and 14, to give some restriction on the integers b_{ij} .

We start with the highest weight vectors $v_{(7,5)}, v'_{(6,6)}, v''_{(6,6)}$ from (26), (27), and (28). The trace polynomial $v_{(7,5)}$ is of the form

$$(34) \quad v_{(7,5)} = -24u_{50}u_{80} + 4u_{60}u_{70} + \dots,$$

where \dots stays for the linear combination of products of the generators (31) with coefficients which are polynomials in S , do not depend on u_{60}, u_{62} , and are without constant term (i.e., from the augmentation ideal $\omega(S)$ of S). By Lemma 2.1, the GL_2 -module generated by $v_{(7,5)}$ has a basis

$$\left\{ v_{(7,5)}, \frac{1}{2}\Delta_1(v_{(7,5)}), \frac{1}{2}\Delta_1^2(v_{(7,5)}) \right\}.$$

Direct computations show that

$$\begin{aligned} \Delta_1(v_{(7,5)}) &= -24(\Delta_1(u_{50})u_{80} + u_{50}\Delta_1(u_{80})) \\ &\quad + 4(\Delta_1(u_{60})u_{70} + u_{60}\Delta_1(u_{70}) + \dots, \end{aligned}$$

and, since $\Delta_1(u_{70}) = 0$,

$$(35) \quad \frac{1}{2}\Delta_1(v_{(7,5)}) = -24(u_{51}u_{80} + u_{50}u_{81}) + 4u_{61}u_{70} + \dots.$$

Similarly

$$(36) \quad \frac{1}{2}\Delta_1^2(v_{(7,5)}) = -48u_{51}u_{81} + 4u_{62}u_{70} + \cdots.$$

The equations (34), (35), and (36) imply that

$$(37) \quad \begin{aligned} v_{(7,5)} &\equiv -24u_{50}u_{80}, \\ \frac{1}{2}\Delta_1(v_{(7,5)}) &\equiv -24(u_{51}u_{80} + u_{50}u_{81}) + 4u_{61}u_{70}, \\ \frac{1}{2}\Delta_1^2(v_{(7,5)}) &\equiv -48u_{51}u_{81} \end{aligned}$$

modulo $\omega(S)B$. In the same way, $v'_{(6,6)}, v''_{(6,6)}$ generate one-dimensional GL_2 -modules isomorphic to $W(6, 6)$ and can be written in the form

$$(38) \quad \begin{aligned} v'_{(6,6)} &\equiv -6(u_{50}u_{81} - u_{51}u_{80}), \\ v''_{(6,6)} &\equiv 108u_{61}^2 + 24u_{70}^2 \end{aligned}$$

modulo $\omega(S)B$. We order the trace polynomials from (31) by

$$(39) \quad u_{50} \succ u_{51} \succ u_{70} \succ u_{80} \succ u_{81} \succ u_{61} \succ u_{90} \succ \cdots \succ u_{12,0},$$

i.e., $u_{i_1j_1} \succ u_{i_2j_2}$ if $i_1 < i_2$ or $i_1 = i_2, j_1 < j_2$, except the case $u_{70} \succ u_{80} \succ u_{81} \succ u_{61}$. Then we extend the order lexicographically on the products of (31). Hence, (37) and (38) give five relations such that, modulo $\omega(S)B$, their leading monomials are

$$(40) \quad u_{50}u_{80}, \quad u_{50}u_{81}, \quad u_{51}u_{80}, \quad u_{51}u_{81}, \quad u_{70}^2.$$

The defining relations of degree 13 and 14 which have been found in the same way as the defining relations of degree 12 show that the corresponding highest weight vectors are of the form

$$(41) \quad \begin{aligned} v_{(8,5)} &= u_{50}u_{90} - u_{60}u_{80} + \cdots, \\ v'_{(7,6)} &= 4(u_{50}u_{91} - u_{51}u_{90}) + (u_{60}u_{81} - u_{61}u_{80}) + \cdots, \\ v''_{(7,6)} &= u_{50}u_{10,0} - 2u_{70}u_{80} + \cdots \end{aligned}$$

$$\begin{aligned}
(42) \quad & v'_{(9,5)} = 2u_{50}u_{11,0} - u_{60}u_{90} + \cdots, \\
& v''_{(9,5)} = \cdots, \\
& v'_{(8,6)} = u_{60}u_{10,0} - 2u_{70}u_{90} + \cdots, \\
& v''_{(8,6)} = u_{50}u_{11,1} - u_{51}u_{11,0} + u_{60}u_{91} - u_{61}u_{90} + \cdots, \\
& v'''_{(8,6)} = -7u_{60}u_{10,0} + 12u_{70}u_{90} + 12u_{80}^2 + \cdots, \\
& v_{(8,6)}^{(4)} = \cdots, \quad v_{(8,6)}^{(5)} = \cdots, \quad v_{(8,6)}^6 = \cdots, \\
& v'_{(7,7)} = -6(u_{60}u_{92} - 2u_{61}u_{91} + u_{62}u_{90}) + u_{70}u_{10,0} + \cdots, \\
& v''_{(7,7)} = \cdots,
\end{aligned}$$

with the same meaning of \cdots as above. Applying several times the derivation Δ_1 on the highest weight vectors from (41) and (42) we obtain that they generate irreducible GL_2 -modules with bases which, modulo $\omega(S)B$, have leading monomials of the form

$$\begin{aligned}
(43) \quad & u_{50}u_{90}, \quad u_{50}u_{91}, \quad u_{51}u_{90}, \quad u_{50}u_{92}, \quad u_{51}u_{91}, \quad u_{51}u_{92}, \\
& u_{50}u_{10,0}, \quad u_{51}u_{10,0}.
\end{aligned}$$

The leading monomials in the case of degree 14 are

$$\begin{aligned}
(44) \quad & u_{50}u_{11,0}, \quad u_{50}u_{11,1}, \quad u_{51}u_{11,0}, \quad u_{50}u_{11,2}, \\
& u_{51}u_{11,1}, \quad u_{50}u_{11,3}, \quad u_{51}u_{11,2}, \quad u_{51}u_{11,3}, \\
& u_{70}u_{90}, \quad u_{70}u_{91}, \quad u_{70}u_{92}, \\
& u_{70}u_{10,0}, \quad u_{80}^2, \quad u_{80}u_{81}, \quad u_{81}^2.
\end{aligned}$$

Theorem 4.1. *Let us fix the homogeneous system of parameters (30) of C_0 , complete it to a system of generators by (31), and let S be defined in (32). The finitely generated free S -module C_0 has a basis of the form (33) such the products in B do not contain factors $u_{i_1j_1}u_{i_2j_2}$ from the lists given in (40), (43), (44).*

Proof. We order the elements (31) by (39). The leading monomials of the defining relations of degree 12, 13, and 14 are given in (40), (43), and (44). If the free generating set contains a monomial from these

lists, we can replace it by a linear combination of monomials which are lower in the lexicographic order and monomials from $\omega(S)C_0$. \square

Remark 4.2. The generating function of the leading monomials from (40), (43), and (44) is equal to $L(t, u) = L_{12} + L_{13} + L_{14}$, where

$$\begin{aligned} L_{12}(t, u) &= t^7u^5 + 3t^6u^6 + t^5u^7 = S_{(7,5)}(t, u) + 2S_{(6,6)}(t, u), \\ L_{13}(t, u) &= t^8u^5 + 3t^7u^6 + 3t^6u^7 + t^5u^8 = S_{(8,5)}(t, u) + 2S_{(7,6)}(t, u), \\ L_{14}(t, u) &= t^9u^5 + 4t^8u^6 + 5t^7u^7 + 4t^6u^8 + t^5u^9 \\ &= S_{(9,5)}(t, u) + 3S_{(8,6)}(t, u) + S_{(7,7)}(t, u). \end{aligned}$$

Comparing with the Hilbert series $H(R_i, t, u)$ of the defining relations R_i of degree $i = 12, 13, 14$, from (8), respectively, we see that $L_{12}(t, u) = H(R_{12}, t, u)$, $L_{13}(t, u) = H(R_{13}, t, u)$, and

$$H(R_{14}, t, u) - L_{14}(t, u) = S_{(9,5)}(t, u) + 3S_{(8,6)}(t, u) + S_{(7,7)}(t, u).$$

The explanation is the following. We multiply the trace polynomials

$$v_{(7,5)}, \quad \frac{1}{2}\Delta_1(v_{(7,5)}), \quad \frac{1}{2}\Delta_1^2(v_{(7,5)}), \quad v'_{(6,6)}, \quad v''_{(6,6)}$$

of degree 12 by the polynomials $u_{10}, u_{11} \in W_1 = W(2, 0) \subset G_0$ and obtain linearly independent relations of degree 14 with generating function which turns to be equal to the difference $H(R_{14}, t, u) - L_{14}(t, u)$. Hence the new relations of degree 14, which cannot be obtained from relations of lower degree, form a GL_2 -module isomorphic to

$$W(9, 5) \oplus 3W(8, 6) \oplus W(7, 7).$$

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